

Analytical Solution Developed For Predicting Transient Interface in Coastal Unconfined Aquifers

Mohamed M. A. Somaida¹(Faculty of Engineering, Port Said University)

Abstract:

For predicting fresh-salt water interface in unconfined coastal aquifers, an equation has been developed to estimate the advance of interface which depends on; the densities of fresh and salt water, the hydraulic properties of the aquifer and the flow velocity components at the boundaries of the aquifer. The developed equation is tried in Sidi Kreir Locality, 32 km west of Alexandria (north-western coastal zone), constituting a coastal unconfined aquifer. The values for advance of interface estimated by the developed equation are compared with those computed using Ghyben-Herzberg relation and are found to have an order of magnitude agreement and are of nearer values. The developed equation is considered advantageous because it considers the hydraulic properties of the aquifer as well as the laws governing the potential flow of two liquids in the porous medium. Assessment of the goodness of the fitting of the compared values for advance of interface shows that; the analytical solution developed, seems to be successful procedure for predicting the movement of the fresh-salt water interface in coastal unconfined aquifers existing in favorable areas.

Keywords : Coastal unconfined aquifer-Water table- Predicting- Transient Interface- Velocity Potential- Goodness of Fitting

1. Introduction

In long-shore areas where permeable materials extend to considerable depth, salt water will underlie fresh water according to Ghyben-Herzberg relation, /1,2/, and will extend downward from the interface to the base of the impermeable material, Fig (1).

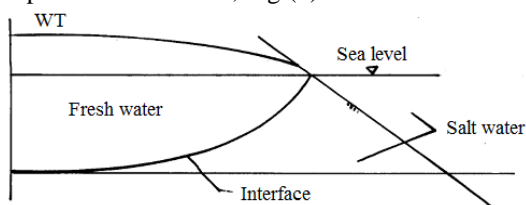


Figure 1: Definition sketch showing the fresh-salt water interface in a coastal unconfined aquifer.

The fresh-water body assumes the form of a lens floating on the underlying sea water and is usually thickest at the central part of the land mass tapering to a thin edge at the shore margins, Fig (1). The flow moves from the thickest part of the lens outward to the shore line and other outlet margins. The magnitude of the fresh-water lens depends mainly on; land mass

width, permeability, recharge and discharge and tidal range /3/.

2. Background

In unconfined aquifers, when the water table drops, there will be a tendency for the fresh-salt water interface to rise. As water levels continue to decline, the interface will continue to advance until conditions of equilibrium are established.

Many studies have been made concerning with groundwater flow in coastal areas. Simple but practical one-dimensional steady-state analytical models, /4,5/, have been developed based on the Ghyben-Herzberg relation and Dupuit's assumptions. Pinder and Cooper (1970), /6/, developed a numerical model for predicting movement of salt-water front in coastal areas. Two-dimensional solutions have been also developed by Van Deer Veer, /7/. More recently, for homogeneous and isotropic aquifers with straight boundaries, analytical solutions can be used to estimate the steady location for the sharp interface /8,9,10/. However the problem of unsteady (transient) interface for coastal unconfined aquifer will be studied beyond the scope here assuming flow velocity potentials in the porous medium and the natural hydrologic

where w_{2o} = vertical upward velocity of movement in salt-water region at $z = 0$. Similarly, for fresh-water region :

$$\left(\frac{du_1}{dx}\right) + \left(\frac{dw_1}{dz}\right) = 0$$

$$\int_{w_1}^{w_{1o}} dw_1 = -\frac{du_1}{dx} \int_L^{z_1} dz \text{ or}$$

$$w_1 = -(z_1 - L)\Phi_{1xx} + w_{1o} \quad (10)$$

Where w_{1o} = velocity of water moving downward vertically through the zone of aeration, resulting from the interaction between the porous medium and the liquid/11/and z_{if} = elevation of free surface from datum, Fig(2)

(5) At free surface pressure (p) = 0, then according to equation (1), $z_1 = -\left(\frac{\phi_1}{K_f}\right)$, accordingly equation (10) will take the form :

$$w_1 = \left(\frac{\phi_1}{K_f} + L\right)\Phi_{1xx} + w_{1o} \quad (11)$$

Put $u_1 = \Phi_{1x}$ and w_1 expressed by equation(11) in equation(5) yields : $n\left(\frac{\partial L}{\partial t}\right) + \left(\frac{\partial L}{\partial x}\right)\Phi_{1x} + \left(\frac{\phi_1}{K_f} + L\right)\Phi_{2xx} - w_{1o} = 0$ (12)

Put $u_2 = \Phi_{2x}$ and w_2 expressed by equation (9) in equation(6) yields: $n\left(\frac{\partial L}{\partial t}\right) + \left(\frac{\partial L}{\partial x}\right)\Phi_{2x} + L\Phi_{2xx} - w_{2o} = 0$ (13)

(6) Combining equations (12) and (13) into a single equation by subtraction, then :

$$2n\left(\frac{\partial L}{\partial t}\right) + (\Phi_{1x} + \Phi_{2x})\left(\frac{\partial L}{\partial x}\right) + \left(\frac{\phi_1}{K_f} + 2L\right)\Phi_{2xx} - (w_{1o} + w_{2o}) = 0 \quad (14)$$

To simplify the continuation of the present mathematical procedure, the term Φ_{2xx} which is the second derivative of the velocity potential related to the salt-water region is usually of infinitesimal value in coastal unconfined aquifers ($2.6 \times 10^{-5} K_s$) in the present study. Hence, the term Φ_{2xx} can be safely eliminated from equation(14) which will now take the following simpler form :

$$2n\left(\frac{\partial L}{\partial t}\right) + (\Phi_{1x} + \Phi_{2x})\left(\frac{\partial L}{\partial x}\right) - (w_{1o} + w_{2o}) = 0 \quad (15)$$

(6) From equation(4), $L = A_1 \Phi_1 - A_2 \Phi_2$, differentiate with respect to x :

$\left(\frac{\partial L}{\partial x}\right) = A_1 \Phi_{1x} - A_2 \Phi_{2x}$, Substitute in (15), then :

$$2n\left(\frac{\partial L}{\partial t}\right) + (\Phi_{1x} + \Phi_{2x})(A_1 \Phi_{1x} - A_2 \Phi_{2x}) = (w_{1o} + w_{2o}) \quad (16)$$

Expansion of brackets and simplifying equation(16) yields :

$$2n\left(\frac{\partial L}{\partial t}\right) + A_1(\Phi_{1x})^2 + A_2(\Phi_{2x})^2 + (A_1 - A_2)\Phi_{1x}\Phi_{2x} = (w_{1o} + w_{2o}) \quad (17)$$

(7) From equation(3), $\left(\frac{\phi_1}{w_s K_f}\right) = \left(\frac{\phi_2}{w_f K_s}\right) + \left(\frac{z}{w_f}\right) - \left(\frac{z}{w_s}\right)$, this equation can be safely approximated since, w_f and w_s (densities) of fresh and sea water are of nearer values :

$\Phi_2 = \frac{w_f}{w_s} \Phi_1$ or $\Phi_2 = A_3 \Phi_1$, However equation(17) will be :

$$2n\left(\frac{\partial L}{\partial t}\right) + (A_1 + A_2(A_3)^2 + A_1 A_3 - A_2 A_3)(\Phi_{1x})^2 = (w_{1o} + w_{2o}) \text{ or}$$

$$2n\left(\frac{\partial L}{\partial t}\right) + A(\Phi_{1x})^2 = (w_{1o} + w_{2o}) \quad (19)$$

Where the parameter A is a constant $(A_1 + A_2(A_3)^2 + A_1 A_3 - A_2 A_3)$ (20)

In equation(19), put $(u_1)^2 = (\Phi_{1x})^2$, where u_1 = horizontal velocity of flow in the fresh-water region, then :

$$\left(\frac{\partial L}{\partial t}\right) = \left(\frac{(w_{2o} + w_{1o}) - A u_1^2}{2n}\right) \quad (21)$$

In equation (21), using the total derivative instead of partial derivative and integrating between the limits, at $t=0, z=L$ and at $t=t, z=L'$, Fig (2), then:

$$\int_L^{L'} dL = \left(\frac{(w_{2o} + w_{1o}) - A u_1^2}{2n}\right) \int_0^t dt \text{ or}$$

$$\Delta L = \left(\frac{(w_{2o} + w_{1o}) - A u_1^2}{2n}\right) t \quad (22)$$

Where ΔL = the length of advance of interface in a time t . If the specific gravity of sea water is proposed 1.025 and that of fresh water = 1.0 and assuming that the fresh and salt-water regions have equal hydraulic conductivities ($K_f = K_s = K$), then the first and fourth terms of the parameter A of equation(20) are equal and eliminates each other, while the second and third terms in the same parameter A, have the same magnitude $\left(\frac{39}{K}\right)$, and can be added such that equation (22) will take the final form :

$$\Delta L = \left(\frac{K(w_{2o} + w_{1o}) - 78 u_1^2}{2nK}\right) t \quad (23)$$

where K = hydraulic conductivity of the aquifer. Equation (23) can be employed to estimate the seasonal rise of interface (ΔL) in a coastal unconfined aquifer. Derivation of equation (23) is based on the following assumptions :

- (1) The aquifer is homogeneous and isotropic and unconfined.
- (2) Two-dimensional potential flow in porous medium.
- (3) The interface between salt and fresh-water is quiet sharp and the transition zone is ignored.

(4) The initial interface is taken at the time where the water table attains its maximum levels and is determined according to Ghyben-Herzberg relation.

(5) The rate of change of the slope of interface is very little and negligible.

(6) The rise of interface is caused by the subsurface outflow of fresh groundwater to the underlying salt base

(7) The fresh and salt-water regions have equal hydraulic conductivity coefficients.

(8) The specific gravity of sea water is proposed 1.025 and that of fresh water is 1.0

(9) The tidal effects are ignored.

4. Application Example

The equation developed for estimating the seasonal advance of interface is applied on the coastal unconfined aquifer in Sidi Kreir Locality, 32 km west of Alexandria, the north western coastal zone, Egypt, Fig(3), /12/.

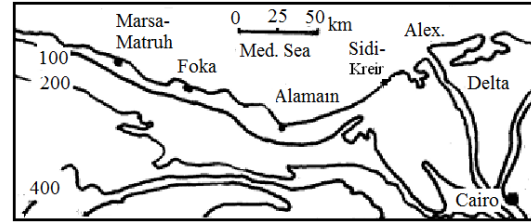


Figure 3: Index map showing the location of Sidi-Kreir locality north western coastal zone, Egypt.

The aquifer consists of permeable carbonate rocks of Pleistocene age, . The aquifer is recharged from rainfall, a large part of it percolates downward to form a fresh groundwater-lens floating on top of the underlying sea water according to Ghyben-Herzberg relation, Fig (4).

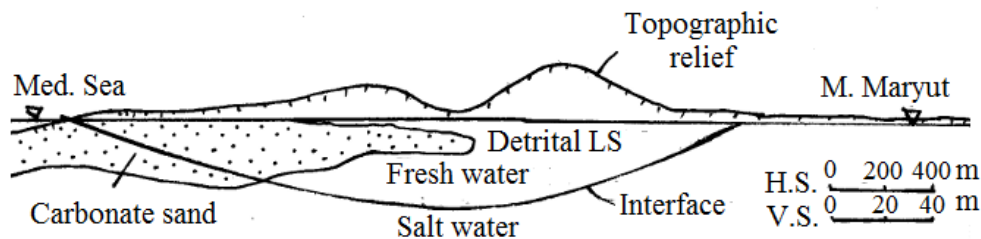


Figure 4: Cross section showing the fresh water lens floating on top of sea water in Sidi-Kreir locality, /12/

The thickness of the aquifer depends on the amount of recharge to the aquifer and the level of water table above sea level. The studied aquifer has an average porosity of 0.385 from core-analysis and an average hydraulic conductivity of 9.4 m/day from pumping tests, /12,13/.

The present aquifer discharges mainly in the Mediterranean sea and Mallahet Maryut. The general direction of groundwater movement is northeast ward north the groundwater divide and southwest ward south the groundwater divide, Figs (5,6).

According to /12/, the long-term studies of water-table fluctuations in Sidi Kreir, shows that, the water levels attain the highest values in February, Fig(5) and the lowest levels in November, Fig(6). These levels are accepted as present-day representative levels. However, the seasonal period of decline of water table is considered 9 months (270 day).

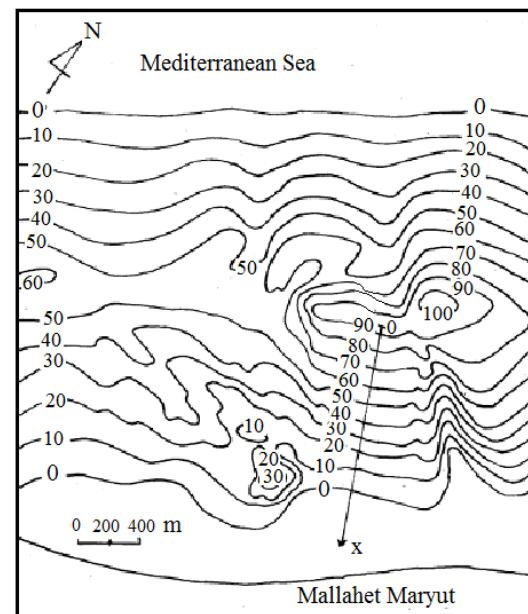


Figure 5: Highest level water-table contour map at Sidi-Kreir locality (February), /12/

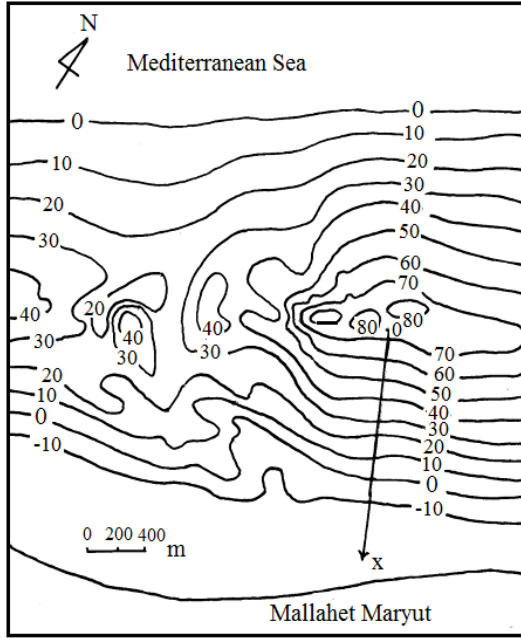


Figure 5: Lowest level water-table contour map at Sidi-Kreir locality (November), /12/

Application of the developed equation on the studied aquifer is made in the following steps :

- (1) The initial interface at $t = 0$, is determined by the use of the water –table contour map of Fig(5) and Ghyben-Herzberg principle
- (2) Datum to which the depth of water table or interface are referred is the horizontal plane touching the apex of the interface. In depth it is about 40 m below sea level, /12/
- (3) The x-direction is represented by a profile extending down gradient from the center of the lens to the south parallel to the direction of groundwater movement to Mallahet Maryut, Figs(5,6). Calculations are tried on selected points with x coordinates from the origin as given in Table(1)
- (4) The horizontal velocity component u_1 , is estimated using the slope of water table in Fig(5) at every selected point and multiplying it by the hydraulic conductivity of the aquifer (9.4 m/day). The results of u_1 are given in Table (1)

5. Calculations and Results

Table 1: Computed hydrologic parameters necessary for application of the developed equation on the studied aquifer in Sidi Kreir Locality

x (m)	Initial elev. of WT (m)	Thickness of aquifer (m)	S (HG)	u_1 (m/day)	Final elev. Of WT (m)	Drop of WT, $\Delta H(m)$	W_{20} (m/day)	W_{10} (m/day)
0.0	1.025	42	0.000938	0.008817	0.8	0.225	0.03333	0.03318
53.3	0.9	36.9	0.000938	0.008817	0.7	0.2	0.02963	0.02942
160	0.79	32.8	0.000833	0.00783	0.6	0.19	0.02852	0.02830
293.3	0.7	28.7	0.000752	0.00707	0.53	0.17	0.02519	0.02498
426.7	0.6	24.6	0.000833	0.007833	0.43	0.165	0.02444	0.02429
533.3	0.5	20.5	0.000938	0.008817	0.34	0.16	0.0237	0.02349
640	0.4	16.4	0.000937	0.008808	0.25	0.15	0.02222	0.02053
746.7	0.3	12.3	0.000938	0.008817	0.16	0.14	0.02074	0.02201
853.3	0.2	8.2	0.000081	0.00861	0.1	0.10	0.01481	0.0146
965	0.087	4.1	0.00075	0.00705	0.0	0.087	0.0129	0.01269
1120	0.0	0.0	0.000645	0.006063	-0.02	0.02	0.00396	0.00275

(5) The vertical upward velocity of salt-water front w_{20} at $z = 0$, at every selected point , is evaluated according to the boundary condition; at $x \geq 0, z = 0, w_{20} = w_2$. Knowing the water-table level drops through 9 months (ΔH) at the selected points, Table(1), then w_{20} is computed using Ghyben-Herzberg relation where the results are shown in Table (1).

(6) For evaluation of the vertical downward velocity in the aeration zone w_{10} , use is made of

the two flow nets constructed for the studied aquifer, one before the rain season (November) and the other after the global rainfall in (February), /14/, where quantitative analysis of the flow nets shows that, the amount of recharge added to the aquifer is 25.4 mm, /14/. This means that, the substantial recharge begins with the rain in November and by February, the water table attains its maximum level. The strata between the land surface and water table have drained 25.4 mm of recharge to move downward

and reach the zone of saturation in 4 months. Hence, the possible average rate of downward travel of groundwater in the zone of aeration may be $(25.4/4 = 6.35 \text{ mm per month})$ or w_{10} averages (0.000212 m/day) . This is an acceptable rate, because it is evaluated according to the actual records and the natural hydrologic conditions of the study area. It must be noted that this velocity will have a negative value because it is oriented in the negative direction of z .

(7) Finally, all the estimated parameters given in Table(1) , are used to apply the developed analytical solution to determine the seasonal advance of interface (ΔL) after 9 months as illustrated in Table(2). For comparison purposes, Table(2) includes also, the corresponding values of (ΔL) estimated using Ghyben-Herzberg relation.

Table 2: Results of advance of interface (ΔL) using the Analytical solution (AS) and Ghyben-Herzberg relation (GHR) for the studied aquifer

Coordinate x (m)	ΔL (AS) (m)	ΔL (GHR) (m)	Coordinate x (m)	ΔL (AS) (m)	ΔL (GHR) (m)
0.0	11.4	9	640	7.5	6
53.3	10.1	8	746.7	7	5.6
160	9.76	7.7	853.3	4.9	4
293.3	8.62	6.8	965	4.3	3.48
426.7	8.35	6.6	1120	0.86	0.8
533.3	8.02	6.4			

6. Analysis of The Results and Conclusions

Correlation of the results of seasonal advance of interface (ΔL) in the studied aquifer given in

Table (2) and their plot in Fig (7), indicates the following :

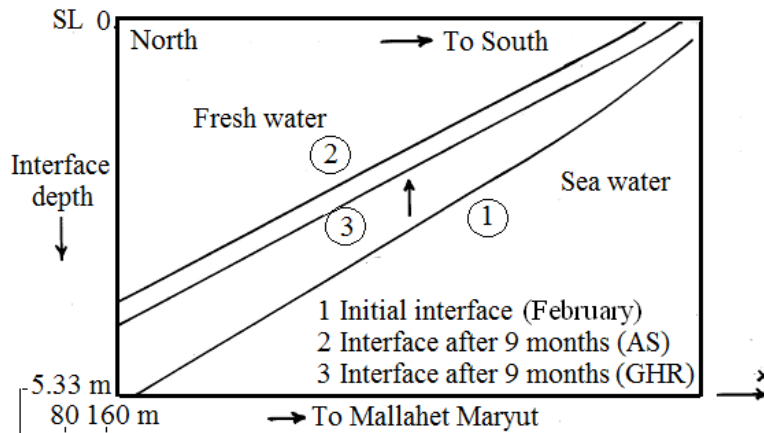


Figure 7: Plot of initial interface (February) in Sidi-Kreir locality and the corresponding plots after 9 months using analytical solution (AS) and Ghyben-Herzberg relation (GHR).

(1)The values of ΔL computed by the developed analytical solution (AS) and those by Ghyben-Herzberg relation (GHR) are found to have an order of magnitude agreement . It is not necessary that the compared values must be exactly equal, because the analytical solution is developed for a potential flow through a porous medium, while the Ghyben-Herzberg relation indicates successive periodical static conditions

(2) The values of ΔL calculated by the analytical solution , are larger than those estimated by the Ghyben-Herzberg relation. The difference in magnitude ranges from 0.06 to 2.4m,(concluded from Table(2), being maximum at the groundwater divide and smaller at the shore line of Mallahet Maryut. This is attributed to the gradual reduction in the thickness of the fresh-water aquifer down gradient towards the Mallaha by subsurface

outflow of fresh-groundwater to the underlying salt base. On the average, the difference in ΔL is 1.23 m, which resembles to an average drop of the water table of 3.075 cm in 9 months or about 0.114 mm/day.

(3) The agreement of the developed analytical solution is considered satisfactory, because the solution considers the natural hydrological conditions of the area as well as the laws governing the potential flow of two liquids in porous medium, these are being out of consideration in the Ghyben-Herzberg principle.

(4) Assessment of the goodness of the fitting of ΔL in both solutions, shows that, the square correlation coefficient (R^2) is 0.999 and the root mean square error (RMSE) is 2.03, /15/. This indicates a strong relation exists between the compared ΔL values, Fig(8).

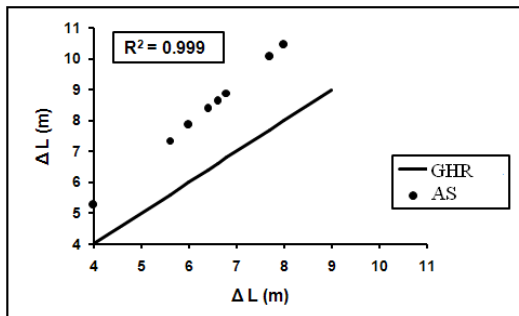


Figure 8: correlation analysis for advance of interface results in Sidi Kreir Locality

(5) Finally, it may be concluded that, the developed analytical solution is a successful procedure for predicting the seasonal advance of interface in coastal unconfined aquifers being found in favorable areas

7. Nomenclature

Φ_1 = Velocity potential for fresh-water region
 Φ_2 = Velocity potential for salt-water region
 K_f = Hydraulic conductivity of fresh-water region
 K_s = hydraulic conductivity of salt-water region
 K = average hydraulic conductivity of the aquifer
 P = pressure along interface at any coordinate x
 w_f = specific weight of fresh water
 w_s = specific weight of salt water
 z = elevation of interface above datum at coordinate x
 A_1, A_2, A_3 = constant coefficients
 L = level of interface at coordinate x
 n = average porosity of the aquifer
 u_1, w_1 = flow velocity components in the fresh-water region of the aquifer in the x and z directions respectively
 u_2, w_2 = flow velocity components in the salt-water region of the aquifer in the x and z directions respectively
 w_{20} = vertical upward velocity of advance of salt water at $z = 0$ at any selected point
 w_{10} = velocity of water movement vertically downward in the aeration zone

Φ_{1x} = rate of change of fresh-water velocity potential with horizontal distance x

Φ_{1xx} = second derivative of fresh-water velocity potential

Φ_{2x} = rate of change of salt-water velocity potential with horizontal distance x

Φ_{2xx} second derivative of salt-water velocity potential

ΔH = seasonal drop of water table

ΔL = seasonal advance of fresh-salt water interface

t = seasonal period of advance

S = slope of water table (hydraulic gradient)

W T = water table

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Author's Information

¹ Professor of Hydraulics in the Department of Civil Eng. , Faculty of Eng., Port-Said University. The Author is very interesting in the fields of, Groundwater Hydrology, Hydraulic Engineering and Applied fluid Mechanics . The author has many publications in these fields (38 papers)